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# DYNAMIC TWINNING PROCESSES IN CRYSTALS

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**Abstract**—This paper describes recent work by the authors on a continuum model for crystal twinning. Twinning is described as an anti-plane shear deformation with discontinuous strains, governed by an elastic potential with multiple wells. Possible shapes of twin lamellae and twinning steps and various regimes of their steady dynamic growth are studied. The model includes a kinetic relation governing anisotropic twin boundary motion in two dimensions under applied stress.

## 1. INTRODUCTION

This paper summarizes some recent studies by the authors on a continuum model for deformation twinning in single crystals. Here we place emphasis on the dynamics of twin growth; we take inertia into account and focus on the kinetics of twin boundaries. The general framework is the continuum theory of phase transitions in thermoelastic crystals (Ericksen, 1984; James, 1981; Abeyaratne and Knowles, 1990, 1991).

In order to model deformation twinning, a non-linear elastic constitutive law was proposed by Rosakis and Tsai (1994) for body-centered cubic (BCC) crystals. This stored energy function possesses multiple potential wells and embodies unstable regimes of shear associated with a failure of ellipticity. The structure of the mechanical response for anti-plane shear deformations is deduced from considerations of lattice symmetry by Tsai (1994) and is consistent with the usual twinning mode in BCC lattices. Inhomogeneous equilibrium deformations of this material involve large, discontinuous shear strains which are localized within narrow twin lamellae. These regions have a shape restricted by metastability. Their boundaries must be closely aligned with special composition planes, have small curvature and must terminate in cusps (Rosakis, 1992). This predicted needle-like configuration is in agreement with observed twin morphology; see, for example, Hull (1964).

We consider the dynamic mechanism of a twin needle growing into a layer. We formulate the problem of steady-state growth of a semi-infinite twin lamella by means of uniform translation of its boundary in the axial direction. Although the component of the velocity normal to the twin boundary is always subsonic (less than the shear wave speed) the steady-state velocity may be subsonic, sonic or supersonic. The full steady-state dynamic problem is solved explicitly in all three cases. In contrast to usual crack and dislocation solutions, the energy remains bounded in the transition from subsonic to supersonic growth. We study the effects of imposing kinetic relations between the driving force on the twin boundary and its normal velocity. Under this additional restriction we find that steady-state subsonic growth cannot occur. However, stress-driven, steady-state growth can happen if the velocity of the lamella tip equals or exceeds the shear wave speed. The kinetic relation determines a critical applied stress for sonic growth; supersonic growth occurs for stresses above this critical value. During such rapid growth we find that shock waves emanate from the tip of the growing lamella.

Our results are in agreement with experimental dynamic studies (Bunshah, 1964; Williams and Reid, 1971) which measure very rapid twin growth speeds, of the order of the shear wave speed. This process is often accompanied by audible sound, known as the twinning cry.

Detailed arguments leading to the results presented here are omitted. The reader is referred to Rosakis and Tsai (1994), Tsai (1994) and Tsai and Rosakis (1994b).

## 2. DYNAMIC ANTI-PLANE SHEAR FOR ANISOTROPIC MATERIALS

A discussion of dynamic anti-plane shear in a three-dimensional context can be found in Tsai and Rosakis (1994a). Consider a body which occupies a cylindrical region $\mathscr{R}$ in its reference configuration. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis with $\mathbf{e}_3$ parallel to the direction of the generators of the region $\mathscr{R}$. Consider motions (time-dependent deformations) $\mathbf{y}(\mathbf{x}, t) = \mathbf{x} + \mathbf{u}(\mathbf{x}, t)$ expressing the current position vector $\mathbf{y}$ at time $t$ of a material point with reference position $\mathbf{x} \in \mathscr{R}$; $\mathbf{u}$ is the displacement vector field. Anti-plane shear motions are ones of the special form

$$\mathbf{y}(\mathbf{x}, t) = \mathbf{x} + \mathbf{u}(\mathbf{x}, t) = \mathbf{x} + u(x_1, x_2, t)\mathbf{e}_3, \tag{1}$$

where $x_i = \mathbf{x} \cdot \mathbf{e}_i$ and $u$ is the scalar out-of-plane displacement field on the cross-section $\Pi$ of $\mathscr{R}$ in the $(x_1, x_2)$ plane. From now on we adopt a two-dimensional description. Greek indices have the range $\{1, 2\}$. A subscript following a comma indicates partial differentiation with respect to the corresponding Cartesian coordinate. Summation over the range of repeated subscribed indices is implied. Time derivatives are dotted. We define the (two-dimensional) shear strain vector $\gamma$ as follows:

$$\gamma = \gamma_\alpha \mathbf{e}_\alpha = u_{,\alpha} \mathbf{e}_\alpha = \nabla u. \tag{2}$$

The displacement $u$ is assumed continuous and piecewise smooth; its gradient $\gamma$ is allowed to jump across certain time-dependent surfaces of strain discontinuity, whose restriction to the cross-section $\Pi$ we denote by $\Gamma_t$. It is a collection of curves on $\Pi$ across which the shear strains $\gamma$ are discontinuous. Their unit normal $\mathbf{n}$ lies in the plane of $\Pi$. These surfaces may move through the body during a dynamic process; their motion is determined by specifying the (scalar) normal velocity $V_n = V_n(\mathbf{x}, t)$ at each point $\mathbf{x}$ of $\Gamma_t$ and time $t$.

We turn to the jump conditions valid on the surfaces of discontinuity. The continuity of $u$ can be shown (Tsai, 1994) to reduce to the following jump conditions on $\Gamma_t$.

$$[\![u_{,\alpha}]\!] l_\alpha = 0, \tag{3}$$

$$[\![\dot{u}]\!] + [\![u_{,\alpha}]\!] n_\alpha V_n = 0, \tag{4}$$

where $l_\alpha$ are components of the unit tangent to $\Gamma_t$.

If certain special restrictions on the constitutive law of the body are met, the full three-dimensional equations of linear momentum balance reduce to a single equation for anti-plane shear motions. This involves only the shear stress components $\sigma_{3\alpha}$ of the nominal (Piola–Kirchhoff) stress tensor and, in addition, $u$ and $\rho > 0$, the constant mass density in the reference configuration:

$$\sigma_{3\alpha,\alpha} = \rho \ddot{u} \qquad \text{on } \Pi - \Gamma_t. \tag{5}$$

For a discussion of the circumstances when this reduction is possible, see Tsai and Rosakis (1994a). This equation of motion is valid away from discontinuity surfaces. It is to be supplemented by a momentum jump condition on $\Gamma_t$:

$$[\![\sigma_{3\alpha}]\!] n_\alpha + \rho [\![\dot{u}]\!] V_n = 0 \qquad \text{on } \Gamma_t. \tag{6}$$

We assume that the body is composed of hyperelastic, compressible material and is homogeneous but anisotropic in the reference configuration. The nominal stress is determined by the constitutive relation as the gradient of the stored energy function. For anti-plane shear, the three-dimensional stored energy function of the deformation gradient tensor reduces to a function $w(\gamma_1, \gamma_2)$ of the shear strains. For our purposes it suffices to consider the shear response of the material, given by

$$\sigma_{3\alpha} = \frac{\partial w}{\partial \gamma_\alpha}(\gamma_1, \gamma_2). \tag{7}$$

The motion of surfaces of strain discontinuity, such as twin boundaries, is accompanied by energy dissipation. A portion of the work done by tractions does not contribute to stored elastic or kinetic energy. The dissipation rate is defined by

$$\Delta(t) = \int_{\partial \Pi} \sigma_{3\alpha} n_\alpha \dot{u} \, ds - \frac{d}{dt} \int_{\Pi} (w + \tfrac{1}{2}\rho \dot{u}^2) \, dA. \tag{8}$$

Abeyaratne and Knowles (1990) showed that

$$\Delta(t) = \int_{\Gamma_t} f V_n \, d_s, \tag{9}$$

where $f$ is called the driving traction acting on the moving surface $\Gamma_t$ and $V_n$ is the normal velocity of the surface. For hyperelastic materials undergoing a dynamic process, the driving traction has a special form obtained by Abeyarante and Knowles (1990). In the anti-plane setting, this specializes to (Tsai, 1994)

$$f = -[\![w]\!] + \tfrac{1}{2}[\![u_{,\alpha}]\!](\sigma^+_{3\alpha} + \sigma^-_{3\alpha}) \qquad \text{on } \Gamma_t. \tag{10}$$

Here the plus and minus superscripts indicate limits as $\Gamma_t$ is approached from its two sides; the unit normal **n** points toward the minus side. The three-dimensional version of this result is reached from a more general theory for thermoelastic materials specialized to isothermal processes (Abeyaratne and Knowles, 1990). The dissipation rate in eqn (9) is equal to the product of the temperature and the entropy production rate, which is required to be non-negative by the Clausius–Duhem version of the second law of thermodynamics. It follows from the localization of eqn (9) that the following dissipation inequality must hold:

$$f V_n \geqslant 0 \qquad \text{on } \Gamma_t. \tag{11}$$

It provides a criterion for restricting the possible directions of the motion of a twin boundary.

## 3. A CONSTITUTIVE MODEL FOR TWINNING

The basic concept underlying twinning is that certain finite shear deformations of a perfect crystal result in a deformed lattice that is identical to the undeformed one, apart from a rotation or reflection. These twinning shears have special crystallographic directions and magnitudes, which can be determined once the original lattice geometry is specified. Twinning involves a planar surface of strain discontinuity (composition plane); the twinning shear deforms the lattice on one side of this plane, while maintaining displacement continuity with the undeformed lattice on the other side.

A continuum mechanical theory of twinning that embodies a notion of crystal symmetry and is capable of predicting twinning modes has been developed by Ericksen (1984), James (1981) and Pitteri (1985). We briefly describe a constitutive model for BCC crystals (Rosakis and Tsai, 1994; Tsai, 1994) which is based on this approach. The twinning shear direction for BCC crystals is of [111] type; here we choose it to coincide with the out-of-plane direction $\mathbf{e}_3$, so that twinning can be described as an anti-plane shear. There are three possible twinning modes with displacements along the given $\mathbf{e}_3$ or [111] direction. Their composition plane normals are of $(11\bar{2})$ type; we arbitrarily choose $\mathbf{e}_2$ of our orthonormal basis to coincide with one of them. The twinning shear vectors corresponding to these three

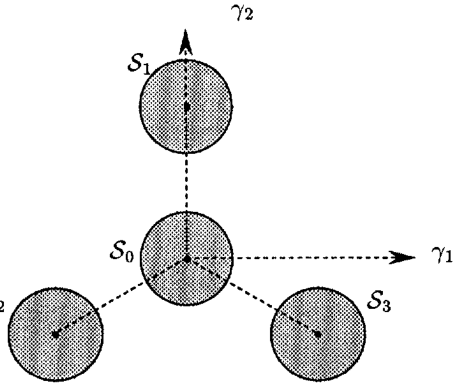


Fig. 1. The strongly elliptic variants $\mathcal{S}_i$ in the shear strain plane.

twinning modes are $\gamma = \boldsymbol{\xi}^i$, $i = 1, 2, 3$. We also set $\boldsymbol{\xi}^0 = \mathbf{0}$. The components of these vectors are given below:

$$(\xi_1^0, \xi_2^0) = (0, 0), \qquad (\xi_1^1, \xi_2^1) = \xi(0, 1),$$
$$(\xi_1^2, \xi_2^2) = \xi(-(\sqrt{3})/2, -1/2), \qquad (\xi_1^3, \xi_2^3) = \xi((\sqrt{3})/2, -1/2), \qquad \xi = (\sqrt{2})/2. \tag{12}$$

The magnitude of the twinning shears is $\xi = 1/\sqrt{2}$. As shown by Rosakis and Tsai (1994), the stored energy function $w$ must have global minima (potential wells) at $\gamma = \boldsymbol{\xi}^i$, $i = 0, 1, 2, 3$. It must be convex (for reasons of stability) in the neighborhood of each well, namely in regions of the $\gamma$ plane of the form

$$\mathcal{S}_i = \{\gamma | \gamma \in \mathbb{R}^2, |\gamma - \boldsymbol{\xi}^i| < \delta\}. \tag{13}$$

We refer to the $\mathcal{S}_i$ as variants; they are disks of radius $\delta < \xi/2$, shown in Fig. 1. The stored energy function $w$ is non-convex for values of strain outside the variants. Thus the property of strong ellipticity fails for strains outside $\mathcal{S}_i$, with a resulting loss of even infinitesimal stability. We assume that the strains always take values in one or more variants. In any deformation that involves twinning, the strains take values in different variants on either side of a twin boundary, hence they must be discontinuous. A specific stored energy function that is consistent with BCC symmetry is given by

$$w(\gamma) = \frac{\mu}{2}|\gamma - \boldsymbol{\xi}^i|^2, \qquad \gamma \in \mathcal{S}_i, \qquad i = 0, 1, 2, 3. \tag{14}$$

The constant $\mu > 0$ is the shear modulus. The values of $w$ for the unstable strains outside $\mathcal{S}_i$ are of no concern to us. This stored energy admits stress-free twinning deformations with discontinuous strain. For example, choose $u = 0$ on one side and $u = \boldsymbol{\xi}^i \cdot \mathbf{x}$ on the other side of a straight line through the origin (composition plane) with unit normal $\mathbf{n} = (\sqrt{}2)\boldsymbol{\xi}^i$. There are three such deformations for $i = 1, 2, 3$, involving different wells of $w$.

## 4. SUBSONIC STEADY TWIN GROWTH

For the constitutive law specified above, the shear stress response function is found from eqns (7) and (14). It is linear in each variant. The equation of motion (5) then reduces to the wave equation

$$\nabla^2 u = \frac{1}{c^2}\ddot{u}, \qquad c = \sqrt{(\mu/\rho)}, \tag{15}$$

valid away from discontinuities. The constant $c = \sqrt{(\mu/\rho)}$ is the shear wave speed. For the special material at hand, it can be shown from the jump conditions (3), (4) and (6) that there are two kinds of travelling discontinuities. In the first case, the strains on either side of $\Gamma_t$ belong to the same variant $\mathscr{S}_i$. Then necessarily the normal velocity $V_n$ must equal the shear wave speed $c$; we refer to these as sonic waves or shocks by an abuse of terminology. They are ordinary elastic shear waves, although the strain jump need not be small. In the second case the strains on either side of $\Gamma_t$ take values in different variants $\mathscr{S}_0$ and $\mathscr{S}_i$. Then the jump conditions (3), (4) and (6) dictate that $V_n < c$. For simplicity we will only consider the variants $\mathscr{S}_0$ and $\mathscr{S}_1$. The jump conditions now reduce to

$$[\![u_{,\alpha}]\!] = [\![\gamma_\alpha]\!] = \frac{\xi n_2 n_\alpha}{1 - V_n^2/c^2}, \tag{16}$$

determining the strain jump across $\Gamma_t$ in terms of the normal velocity $V_n$ and the normal $\mathbf{n}$. Such a $\Gamma_t$ is called a subsonic twin boundary. The linearity of the stress response inside each variant dictates that sonic waves are dissipation free: $f = 0$. In contrast the driving traction on twin boundaries does not vanish in general; one finds from eqn (10)

$$f = \frac{\mu\xi}{2}(u,_2^+ + u,_2^- - \xi) = \frac{\xi}{2}(\sigma_{32}^+ + \sigma_{32}^-). \tag{17}$$

We briefly recall some results from statics; see Rosakis (1992) and Rosakis and Tsai (1994). Suppose that a bounded region $\mathscr{D}$ has twinned in an infinite crystal. The twin boundary is $\Gamma$, a closed curve; the region outside $\mathscr{D}$ is $\mathscr{M}$. Thus

$$\nabla u \in \mathscr{S}_0 \quad \text{on } \mathscr{M}; \qquad \nabla u \in \mathscr{S}_1 \quad \text{on } \mathscr{D}. \tag{18}$$

In the case of equilibrium eqns (15) and (16) reduce to

$$\nabla^2 u = 0 \quad \text{on } \Pi - \Gamma, \qquad [\![u_{,\alpha}]\!] = \xi n_2 n_\alpha \quad \text{on } \Gamma. \tag{19}$$

Assuming $u \to 0$ away from $\mathscr{D}$, the solution to eqn (19) is given in terms of a logarithmic potential of the twin boundary $\Gamma$:

$$u(\mathbf{x}) = u_{\mathscr{D}}(\mathbf{x}) = -\frac{\xi}{2\pi}\int_\Gamma \log|\mathbf{x} - \mathbf{z}| n_2(\mathbf{z})\, ds_z. \tag{20}$$

The displacement gradient must be confined to the appropriate variants by eqn (18). This places severe restrictions on the shape of $\mathscr{D}$ (Rosakis, 1992; Rosakis and Tsai, 1994): the boundary curve $\Gamma$ must have sufficiently small slope (with respect to the composition plane) and curvature; the outward normal $\mathbf{n}$ must be close in direction to the $\mathbf{e}_2$ direction. Hence $\Gamma$ cannot be a smooth curve, such as an ellipse. At the same time, it cannot have corners, which would cause logarithmic singularities in the strains, thus violating eqn (18); hence $\Gamma$ must have cusps. Therefore only slender, flat, needle-like regions are possible, in agreement with experimental observation. An example is the lamella. Such a region can be described

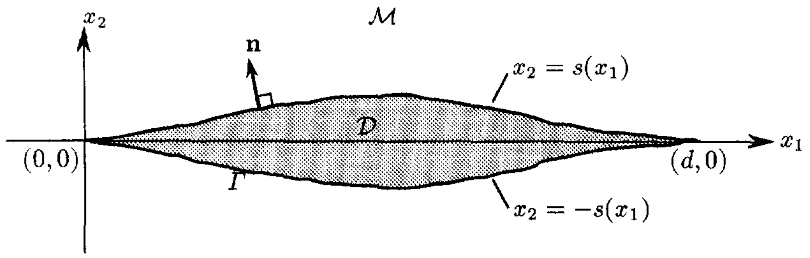


Fig. 2. Shape of a bounded twin lamella.

by $\mathscr{D} = \{\mathbf{x} | -d < x_1 < d, \ -s(x_1) < x_2 < s(x_1)\}$. See Fig 2, where $\mathscr{D}$ is shown shaded. Then the shape curve $s$ must satisfy $s(\pm d) = s'(\pm d) = 0$, so that the endpoints are cusps. Also $s'$ and $s''$ must be sufficiently small in order for eqn (18) to hold, i.e. so that the strains are confined to the stable variants.

A commonly observed twin morphology involves a perfectly flat layer along the $x_1$-axis. See Chu and James (1993) for micrographs of layered microstructures. It is often observed that such layers often terminate inside the crystal at a cusped, needle-like tip. We model this as a semi-infinite lamella shown shaded in Fig. 3. The twin boundary is

$$\Gamma: \quad x_2 = \pm s(x_1), \quad s(x_1) = h = \text{const.} \quad \text{for } x_1 \leqslant 0, \quad s(d) = 0. \tag{21}$$

The region $\mathscr{D}$ is a layer of thickness $2h$ for $x_1 < 0$; it tapers to a point (tip) at $(d, 0)$. The out-of-plane displacement is still of the form (20), although the integral is to be suitably interpreted since the domain of integration is unbounded (Tsai, 1994). This leads to similar restrictions on the slope and curvature of $s$ and the cusped nature of the tip $[s'(d) = s'(0) = 0]$ as in the case of a bounded lamella.

The full dynamic problem involving transient motion and shape changes of the twin boundary is difficult to analyze. Consider instead the following steady-state twin growth problem (Tsai, 1994). A semi-infinite lamella, such as the one shown in Fig 3, is assumed to grow by steady translation in the positive $x_1$ direction with speed $V = \text{const}$. Now $\mathscr{D}_t$, $\mathscr{M}_t$ and $\Gamma_t$ are its time-dependent interior, exterior and boundary, respectively, so that $\Gamma_t$ is described by $x_2 = \pm s(x_1 - Vt)$; see eqn (21). We seek a steady-state solution $u$ of the dynamic equations (15) and (16), of the form

$$u(x_1, x_2, t) = \mathring{u}(x_1 - Vt, x_2). \tag{22}$$

The body is subject to a uniform remotely applied shear stress $\sigma_\infty$ in the twinning direction $\mathbf{e}_2$, so that

$$u_{,2} \to \gamma_\infty = \sigma_\infty / \mu \qquad \text{as } |x_2| \to \infty. \tag{23}$$

Use of eqn (22) reduces eqn (15) to

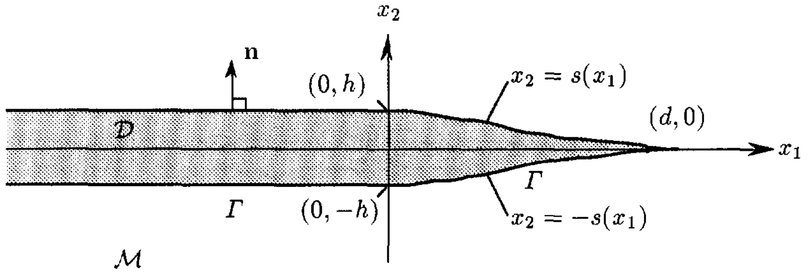


Fig. 3. A semi-infinite twin lamella.

$$\left(1-\frac{V^2}{c^2}\right)\mathring{u}_{,11}+\mathring{u}_{,22}=0. \tag{24}$$

This equation is elliptic, parabolic, or hyperbolic, depending on whether $V$ is less than, equals, or exceeds the shear wave speed $c$. We refer to these three cases as subsonic, sonic and supersonic steady growth, respectively. Since $\Gamma_t$ is a moving twin boundary separating regions with strains in different variants, its normal velocity is subsonic: $V_n < c$. It is related to the steady-state speed $V$ by $V_n = Vn_1$, involving a component of the outward unit normal $\mathbf{n}$. Note that $V$ itself need not be less than $c$; $n_1$ can be arbitrarily small if the tapered position of $\Gamma_t$ is flat enough. Detailed restrictions on $V$, $V_n$ and $\mathbf{n}$, stemming from the strain confinement conditions (18), are given by Tsai (1994).

For the moment, we consider subsonic growth ($V < c$). We employ the Lorentz transformation, namely

$$\mathring{u}(x_1-Vt,x_2)=\overset{*}{u}\left(\frac{x_1-Vt}{\sqrt{[1-(V^2/c^2)]}},x_2\right); \tag{25}$$

here the $x_1$-axis is stretched by the Lorentz ratio $\lambda = 1/\sqrt{[1-(V^2/c^2)]}$. After the transformation, eqns (15) and (16) become

$$\nabla^2\overset{*}{u}=0 \qquad \text{on } \Pi-\overset{*}{\Gamma} \qquad [\![\overset{*}{u}_{,\alpha}]\!]=\xi\overset{*}{n}_\alpha\overset{*}{n}_2 \qquad \text{on } \overset{*}{\Gamma}. \tag{26}$$

Note that $\overset{*}{\mathbf{n}}$ is the outward unit normal to the transformed boundary $\overset{*}{\Gamma}=\partial\overset{*}{\mathscr{D}}$, which is stretched by $\lambda$ along the horizontal direction in the moving coordinates. Observe that eqns (26) are identical in form to the field equation and jump condition (19) governing the *static* problem of a semi-infinite twin lamella discussed above. Using this fact and the transformations (24) and (22), the displacement is found to be

$$u(x_1,x_2,t)=u_{\overset{*}{\mathscr{D}}}(\lambda(x_1-Vt),x_2)+\gamma_\infty x_2, \tag{27}$$

where $u_{\overset{*}{\mathscr{D}}}$ is a logarithmic potential of the transformed region $\overset{*}{\mathscr{D}}$ stretched by the Lorentz factor $\lambda$ but otherwise completely analogous to eqn (20). The second term above is due to the applied shear; see eqn (23). As a result, the shape of the tapered portion of $\Gamma_t$ must be sufficiently flat and terminate in a cusp—as in statics—in order for the strain confinement conditions (18) to be satisfied. Detailed estimates of the strains due to the solution (27) in terms of the shape function $s$ and speed $V$ are found in Tsai (1994).

## 5. KINETICS

Apart from certain restrictions on the shape of the curved portion of the lamella near the tip, the shape function $s$ of $\mathscr{D}$, the propagation speed $V$ and the applied shear stress $\sigma_\infty$ can be specified independently in order to determine the displacement field $u$ in eqn (27). On physical grounds one would expect that the applied stress would at least partially determine the speed of propagation of twin boundaries. The constitutive model so far fails to predict the motion of twin boundaries.

A similar situation is encountered by Abeyaratne and Knowles (1991) in a one-dimensional Riemann problem involving propagation of phase boundaries in elastic bars. They found that the underlying balance laws and the dissipation inequality fail to produce a unique solution. To remedy this situation, they supplement the basic field equations with additional constitutive assumptions: a nucleation criterion which signals the first occurrence of the phase change and a kinetic relation which relates the rate of phase transformation (speed of the phase boundary) to the driving traction. Abeyaratne and Knowles (1991) showed that these traditional criteria select a unique solution to the Riemann problem. Generalizing their approach, we introduce a class of kinetic relations governing the motion of possibly curved twin boundaries in two dimensions. We postulate that there exists a constitutive relation $\tilde{V}_n$, which determines the normal velocity $V_n$ at each point of $\Gamma_t$ in terms of the local driving traction $f$ and, in addition, the orientation of the twin boundary $\Gamma_t$ at that point. Specifically, we postulate

$$V_n = \tilde{V}_n(f, n_1), \tag{28}$$

where $\tilde{V}_n$ is the kinetic response function characteristic of the material. Abeyaratne and Knowles (1990, 1991) have postulated a relation of the form $V_n = \tilde{V}_n(f)$. This is suitable for one-dimensional problems or for isotropic materials. We add an explicit dependence on the local boundary orientation through the normal $\mathbf{n}$, or, equivalently, through $n_1 = \mathbf{n}\cdot\mathbf{e}_1$. This is an attempt to take into account the anisotropy of the material. The kinetic response function $\tilde{V}_n$ is assumed to have the following properties:

$$\begin{aligned}
&\text{(i)} && f\tilde{V}_n(f, n_1) \geqslant 0,\\
&\text{(ii)} && \tilde{V}_n(f, 0) = 0,\\
&\text{(iii)} && \frac{\partial \tilde{V}_n}{\partial f}(f, n_1) > 0,\\
&\text{(iv)} && \frac{\partial \tilde{V}_n}{\partial n_1}(f, n_1) \geqslant 0, \qquad \text{for } n_1 > 0.
\end{aligned} \tag{29}$$

Condition (i) follows from the dissipation inequality [eqn 11]. Consider a perfectly flat twin boundary, so that $\mathbf{n} = \mathbf{e}_2$ ($n_1 = 0$) throughout $\Gamma_t$. Condition (ii) states that such a twin boundary cannot move. This is a strong assumption. It is consistent with the dislocation model of curved twin boundaries. According to this model, the dislocation density on a twin boundary is proportional to $n_1$. Hence no dislocations are present on the flat portions ($n_1 = 0$). In addition, the growth of twins would occur by dislocation glide along the $x_1$ direction. Thus flat portions cannot climb (move along the $x_2$ direction) in this model. Note that this does not rule out the overall growth in the $x_2$ direction. This can occur by nucleation and subsequent motion of twinning steps in the $x_1$ direction. The motion along $x_1$ of a step of height $h$ would result in a translation along $x_2$ of an otherwise flat boundary by a distance $h$. The motion along $x_1$ of needles (lamellae) is still possible under (ii). Such motion of steps and needles appears to be the dominant mechanism for twin growth and has been abundantly observed in experiments (Williams and Reid, 1971; Chu and James, 1993). Assumption (iii) states that an increase in the driving traction increases the normal velocity. Most (one-dimensional) kinetic relations proposed so far abide by (iii) (Porter and Easterling, 1981; Abeyaratne and Knowles, 1990, 1991; Truskinovsky, 1985). Assumption (iv) is consistent with (i) and (ii). For $f > 0$, (i) implies $\tilde{V}_n(f, n_1) \geqslant 0$ for $n_1 > 0$. Together with (ii), this in turn strongly suggests that (iv) should be true. This assumption implies that portions of $\Gamma_t$ with higher "orientation imperfection" (greater $n_1$) are more "unstable" or more mobile under driving traction.

Returning to steady growth of the twin lamella considered above, we note that $V_n$ and $f$ are functions of position along the twin boundary $\Gamma_t$. We now require that the kinetic relation (28) holds pointwise there. Recall eqns (17) and (27). It can be shown (Tsai, 1994) that

$$f = f_\infty + f_{\mathscr{S}}, \qquad f_\infty = \xi\sigma_\infty. \tag{30}$$

Here $f_\infty$ is the (constant) contribution of the applied stress $\sigma_\infty$ to the driving traction, while $f_{\mathscr{S}}$, which varies with position, is the contribution due to $u_{\mathscr{S}}$ in eqn (27) analogous to the self-force of a curved dislocation loop. Recalling that $V_n = Vn_1$ and eqn (21), note that $V_n = 0$ on the flat portion of $\Gamma_t$ ($n_1 = 0$ for $x_1 < 0$), so that eqn (28) holds there because of eqn (29) (ii). The second term $f_{\mathscr{S}}$ in eqn (30) can be expressed as a line integral involving the shape function $s$ in eqn (21). Recalling that the tip $(d, 0)$ must be a cusp and that no corners are possible, we have $s'(0) = s'(d) = 0$. Assuming $s$ to decrease monotonically along the curved position, it can be shown that

$$f_{\mathscr{S}}(0, h) < 0, \qquad f_{\mathscr{S}}(d, 0) > 0; \tag{31}$$

recall that in the moving coordinates, the tip of the lamella is at $(d, 0)$, while $\Gamma_t$ becomes flat at $(0, h)$; see Fig. 3. For details, see Tsai (1994). The geometric restrictions on $s$ imply that there are two points close to $(0, h)$ and to $(d, 0)$ on the curved part of $\Gamma_t$ with the same value of $n_1$, hence the same normal velocity $V_n$. Let the values of $f$ at these two points be $f_1$ and $f_2$. Then eqn (28) dictates that $Vn_1 = \tilde{V}_n(f_1, n_1) = \tilde{V}_n(f_2, n_1)$. However, because of eqns (31) and (30), we have $f_1 < f_2$, so that eqn (29) (iii) is contradicted. We conclude that steady subsonic ($V < c$) growth of lamellae is impossible under a wide class of kinetic relations for which the normal velocity depends monotonically on the driving traction. Note that this observation does not rule out subsonic propagation completely. It merely suggests that such growth cannot be steady; it must be transient and involve change of shape of the tapered part of the lamella. This situation is beyond our present scope. As we show below, our assumptions on kinetics do in fact allow for steady sonic and supersonic propagation.

## 6. SUPERSONIC STEADY TWIN GROWTH

We consider sonic and supersonic growth speeds $V \geqslant c$. The Lorentz transformation [eqn (25)] now fails. Instead, we recall the steady motion equation (24) and the jump conditions (16). The former is parabolic for sonic propagation. For $V = c$ eqns (24) and (16) reduce to

$$\hat{u}_{,22} = 0, \qquad [\![\hat{u}_{,\alpha}]\!] = \xi n_\alpha / n_2. \tag{32}$$

This problem is easily solved, yielding

$$u = (\xi + \gamma_\infty) x_2 \quad \text{on } \mathscr{D}_t, \qquad u = \gamma_\infty x_2 + \operatorname{sgn}(x_2) s(x_1 - ct) \quad \text{on } \mathscr{M}_t. \tag{33}$$

Two important characteristics distinguish this sonic solution from its subsonic counterpart [eqn (27)]. The parabolic nature of eqn (32) allows $s$ in eqn (33) to be merely piecewise smooth, so that corners are allowed for sonic growth. In the subsonic case, corners would be impossible, since they would lead to unbounded strains, which in turn would violate the strain confinement conditions (18). Instead, for sonic growth, each corner on $\Gamma_t$ is accompanied by an additional sonic discontinuity (shock wave) in $\mathscr{M}_t$, parallel to the $x_2$-axis and moving to the right speed $c$. In addition, it follows from eqns (33) and (17), that the driving traction is independent of the shape and constant:

$$f = \xi \sigma_\infty. \tag{34}$$

The tapered part of $\Gamma_t$ can be chosen to be wedge-shaped [Fig. 4(b)], with shape described by

$$s(x_1) = h \quad \text{for } x_1 < 0; \qquad s(x_1) = h(1 - x_1/d) \quad \text{for } 0 \leqslant x_1 \leqslant d. \tag{35}$$

The lamella thickness is $2h$; it then tapers linearly to a point at $(d, 0)$. This tip and the points $(0, \pm h)$ are now corners [Figure 4(b)]; the vertical dotted lines emanating from them are sonic waves (shocks), while the solid lines are twin boundaries. The strains are piecewise constant; they suffer discontinuities across sonic shocks and twin boundaries. Noting that now both $f = \xi\sigma_\infty$ and $n_1 = h/d$ are constant on the inclined portion of $\Gamma_t$, it is clear that the kinetic relation (28) can be satisfied by choosing $\sigma_\infty$ and the slope $h/d$ so that

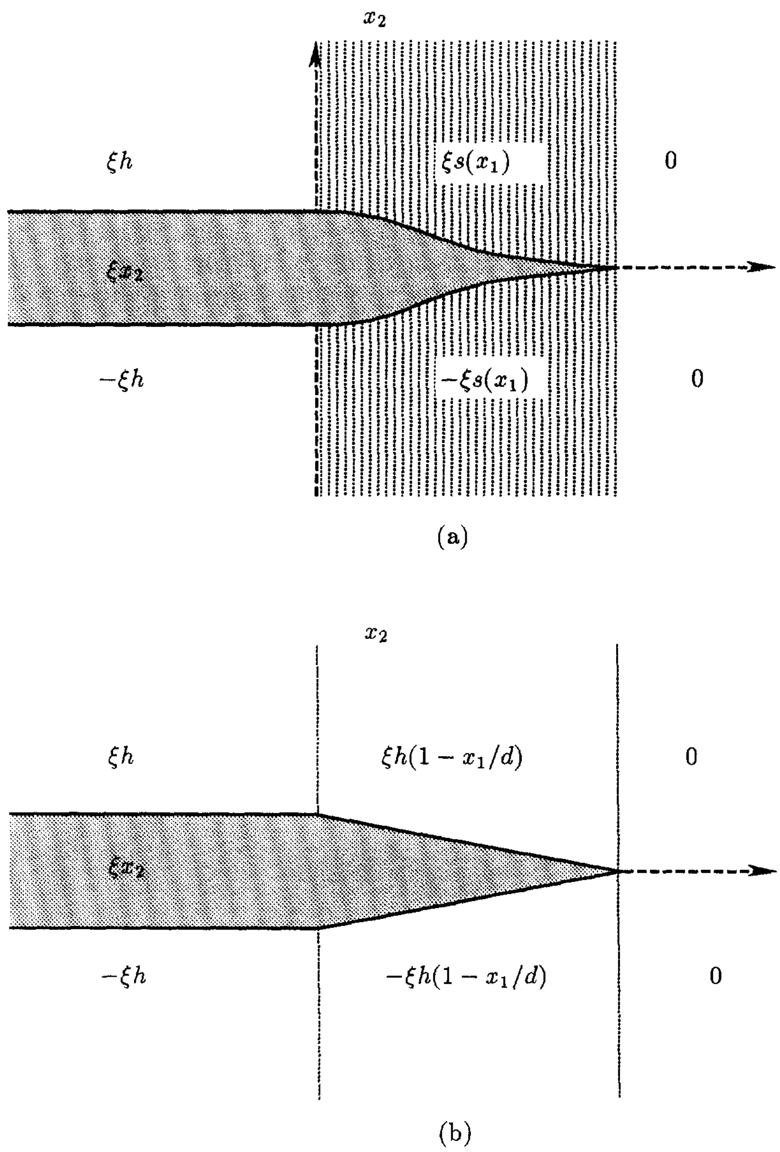


Fig. 4. Displacement fields for sonic propagation, $V = c$. (a) Smooth boundary; dotted lines represent level curves of displacement. (b) Boundary with corners; dotted lines are shock waves (strain discontinuities) emitting from corners.

$$c = \tilde{V}_n(\xi\sigma_\infty, h/d)d/h. \tag{36}$$

Here we recall that $V_n = Vn_1$, $V = c$. As an example, consider the special kinetic relation

$$\tilde{V}_n(f, n_1) = Mf|n_1|. \tag{37}$$

This abides by all restrictions (29). It implies that $V = Mf$, where $M > 0$ is a constant mobility coefficient. Then we find that sonic propagation occurs at a critical value of the applied stress

$$\sigma_\infty = c/M\xi, \tag{38}$$

regardless of the shape of the lamella tip.

Next we consider supersonic growth with $V > c$. Note that the slope of $\Gamma_t$ must be small enough so that $V_n = Vn_1 < c$, since twin boundaries must have subsonic normal velocities. Now equation (24) is hyperbolic and can be solved using the method of characteristics. Letting $\lambda = 1/\sqrt{[(V^2/c^2)-1]}$, the characteristics are straight lines with slope $\pm\lambda$. The solution is of the form

$$u(x_1, x_2) = p(\lambda x_1 - x_2) + q(\lambda x_1 + x_2), \tag{39}$$

where the functions $p$ and $q$ are piecewise smooth and completely determined in terms of the shape function $s$ and the applied stress. Their specific form is rather complicated though explicit; we omit it here. See Tsai (1994) for details. It turns out again that corners are possible; pairs of sonic shocks emanate from each corner along characteristics. For the wedge-tipped lamella [eqn (35)] the shocks are shown as dotted lines in Fig. 5. Together with the twin boundaries (solid lines), they separate $\mathscr{D}_t$ and $\mathscr{M}_t$ into regions, inside each of which the strain is constant, taking the vectorial value displayed in Fig. 5. To these one should add the term $\gamma_\infty \mathbf{e}_2$ due to the applied stress. The supersonic solution reduces to the sonic one [eqn (33)] upon formally setting $V = c$.

Note that the strain confinement conditions (18) still apply in the present circumstances; they lead to a restriction on the slope $h/d$ of the wedge that takes the form

$$0 < h/d < g(V, \sigma_\infty), \qquad V \geqslant c; \tag{40}$$

here $g$ is an explicitly determined but complicated function, given by Tsai (1994). The supersonic solution yields an expression for the driving traction which can be substituted into the kinetic relation (37) to yield

$$V/M = f = \xi\sigma_\infty - \frac{\mu\xi^2}{2}(\lambda d/h - h/d\lambda)^{-1} \qquad \text{for } V > c. \tag{41}$$

Noting that $\lambda = 1/\sqrt{[(V^2/c^2)-1]}$, the above is an implicit relation between $V$, $\sigma_\infty$ and $h/d$, subject to the restriction (40). The applied stresses must be in the interval

$$c/M\xi \leqslant \sigma_\infty < \mu\delta; \tag{42}$$

if $\sigma_\infty = c/M\xi$ then $V = c$ (sonic growth). Supersonic growth occurs for applied stresses

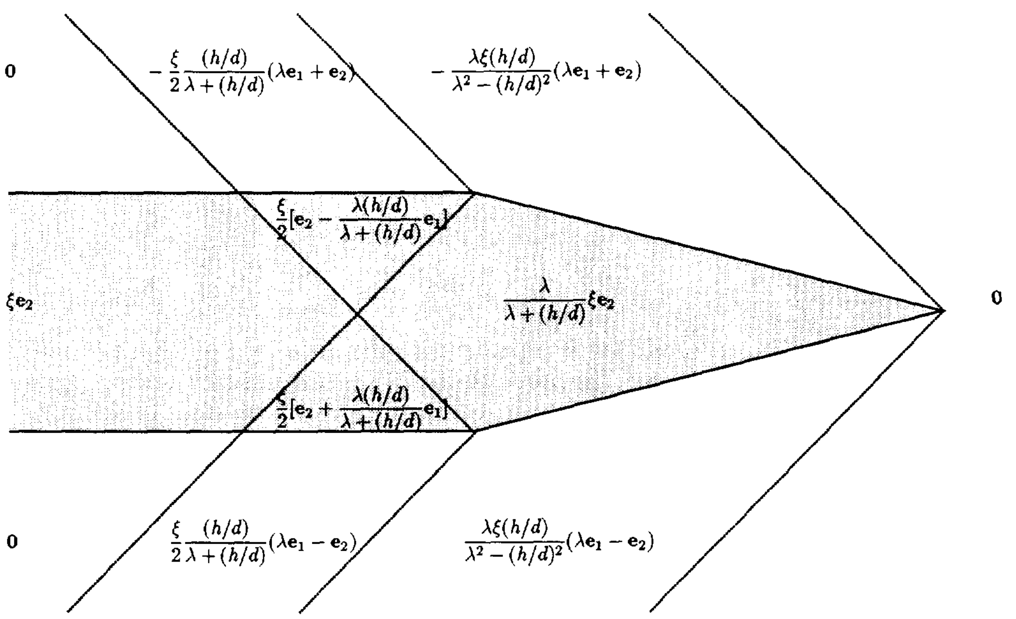


Fig. 5. Shear strains for supersonic growth. Shocks are shown dotted.

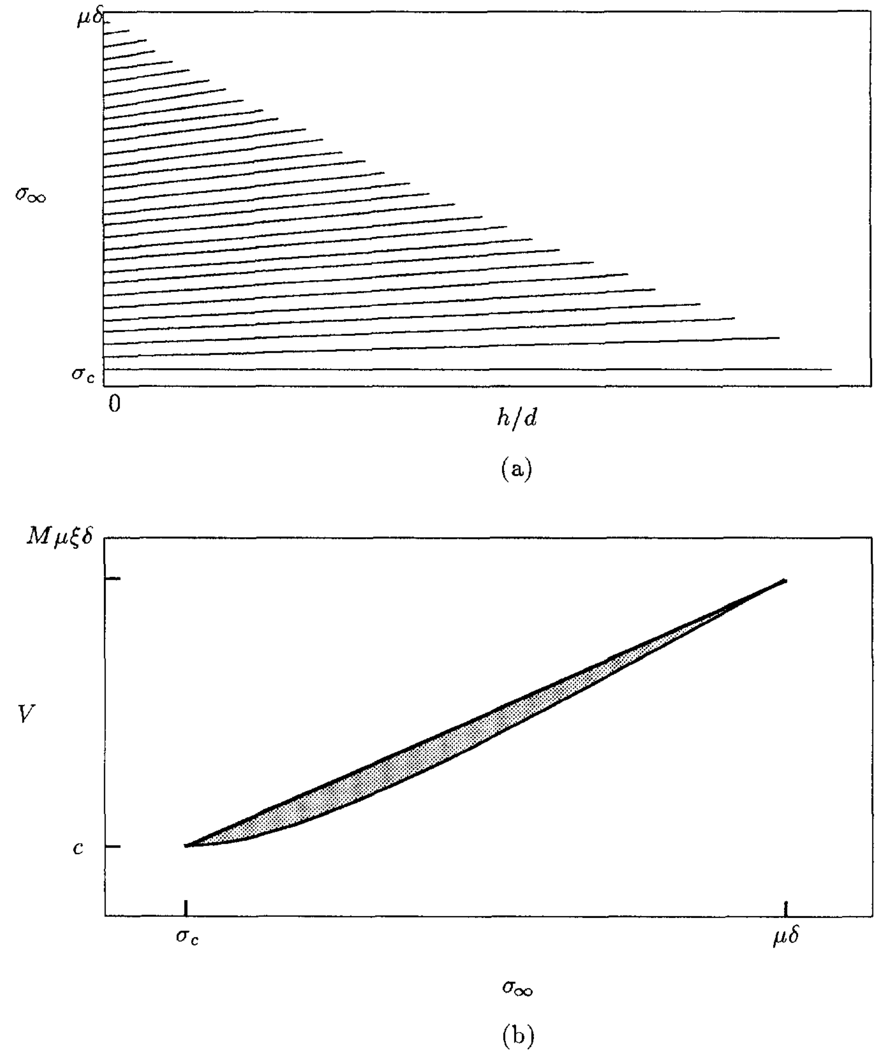


Fig. 6. (a) Constant speed ($V$) line on the ($h/d$, $\sigma_\infty$) plane. (b) The admissible region for the values of $\sigma_\infty$ and $V$, $\sigma_c = c/(M\xi)$ is the critical stress for sonic growth.

above this critical value. Figure 6(a) shows curves of $\sigma_\infty$ versus $h/d$ for various fixed values of $V$; these curves terminate according to the restriction (40). The growth speed $V$ is not completely determined by $\sigma_\infty$; it depends on $h/d$ as well. The region of possible values of $(V, \sigma_\infty)$ dicted by eqns (40) and (41) is shown shaded in Fig. 6(b). For each value of $\sigma_\infty$ there is a small range of possible velocities, so that

$$\phi(\sigma_\infty) < V < M\xi\sigma_\infty. \tag{43}$$

Both the upper and lower bounds for $V$ are monotonically increasing functions of applied stress. The upper bound is linear. The lower bound can be obtained from eqns (37) and (40). These two bounds approach each other near the limiting stress values in eqn (42).

In summary, we find that under a certain class of kinetic relations, subsonic growth of twin needles into layers cannot be steady. Instead, it probably involves transient effects and shape changes. However, if the applied shear stress is large enough, it is possible to have steady growth that conforms to the kinetic relation at tip speeds that equal or exceed that of shear waves. Such sonic or supersonic growth causes the emanation of shock waves, which are likely to be responsible for the audible sound—the cry of twins—often heard during experiments. The growth speed is only partially determined by the applied stress.

However, it is confined between upper and lower bounds that are increasing functions of the applied stress.

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